Generalized Solutions to Degenerate Dynamical Systems

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We consider degenerate dynamical systems, DDSs in short, of the form:

$$A(x)\dot{x} = f(x), \qquad \text{where} \quad x \in \mathbb{R}^n, \quad f \text{ is smooth}, \tag{1}$$

and $x \mapsto A(x)$ is a smooth field of $n \times n$ matrices. The **singular locus** $\mathcal{Z} = \{x \in \mathbb{R}^n;\ \det(A(x)) = 0\}$ is assumed to be non empty but with empty interior.

This work follows a question asked by the Chilean physicist Jorge Zanelli (See [2]): is it possible to classify systems (1) at singular points?

For this purpose we consider (1) as a differential inclusion $\dot{x} \in \Gamma(x)$ in order to get solutions inside or through $\mathcal{Z}$. The main issue consists in correctly defining the sets $\Gamma(x)$ at singular points.

A first attempt imitates Filippov's definition ([3]): The set $\Gamma_f(p)$ is defined at $p \in \mathcal{Z}$ by

$$\Gamma_f(p) = \bigcap_{r>0} \overline{\text{co}}\ \{F(x) = A(x)^{-1} f(x); \quad x \in B(p,r) \setminus \mathcal{Z}\}.$$

It allows to solve (1) and to get solutions inside $\mathcal{Z}$ in simple cases.

However a detailed analysis of generic systems shows that $\Gamma_f(p) = \mathbb{R}^n$ at a generic point $p \in \mathcal{Z}$ of such a system. The problem comes from the fact that in our case the vector field $F(x) = A^{-1}(x) f(x)$ is unbounded though Filippov's theory deals with bounded discontinuous vector fields.

These drawback leads us to a different definition thanks to which $\Gamma(p)$ is, at a generic point $p \in \mathcal{Z}$ of a generic system, a one-dimensional affine space. The direction of this space is given by the vector field:

$$\widetilde{f}(x) = \widetilde{A}(x) f(x), \qquad \text{where} \qquad \widetilde{A}(x) f(x) = (\text{co } A(x))^T.$$

The vector field $\widetilde{f}(x)$ is well defined at all points of $\mathbb{R}^n$, hence of $\mathcal{Z}$, and allows to get the following results:

- The intersection of $\Gamma(p)$ with the tangent space to $\mathcal{Z}$ at p defines a vector field in an open and dense subspace of $\mathcal{Z}$.
- Generalized solutions may enter, leave, or remain in the singular locus.
- The trajectories ending at $\mathcal{Z}$ can be computed by reparametrization.

The result are finally applied to the models of [2].

References

[1] P. Jouan, U. Serres *Generalized solutions to degenerate dynamical systems*, Journal of Mathematical Physics 64, 062703 (2023).

[2] J. Saavedra, R. Troncoso, J. Zanelli *Degenerate dynamical systems*, J.Math.Phys., **42**, 4383 (2001).

[3] A.F. Filippov, *Differential equations with discontinuous right-hand side*, Kluwer (1988).